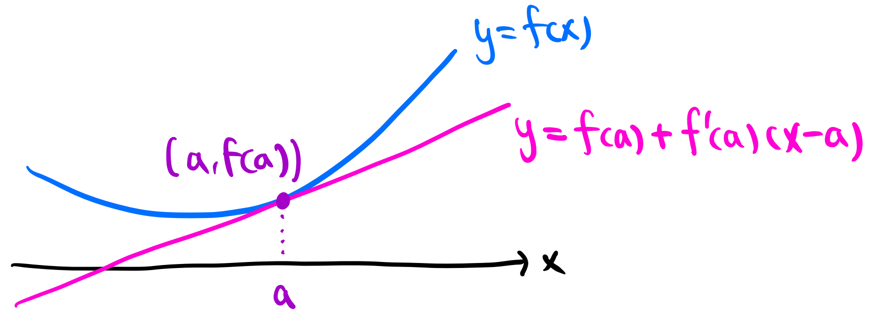
14.4. Tangent planes and linear approximations

Recall : Given a differentiable function $f(x)$, the tangent line to the graph $y=f(x)$ at $x=a$ is given by

$$y = f(a) + f'(a)(x-a)$$

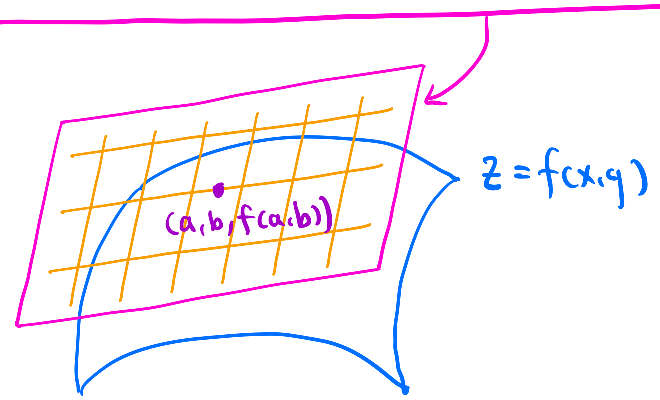


$\Rightarrow$ The linear approximation of $f(x)$ near $x=a$ is

$$f(x) \approx f(a) + f'(a)(x-a)$$

Prop Given a differentiable function $f(x,y)$, the tangent plane to the graph $z=f(x,y)$ at (a,b) is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



$\Rightarrow$ The linear approximation of $f(x,y)$ near (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Note (1) The tangent plane equation can be written as

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z-f(a,b)) = 0.$$

$\Rightarrow$ A normal vector is $\vec{n} = (f_x(a,b), f_y(a,b), -1)$

(2) The tangent plane equation can also be written as

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\Rightarrow dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

where dx, dy, dz are called "differentials", and represent small changes or errors in x, y, z.

Prop Given a differentiable function $f(x,y,z)$, its linear approximation near (a,b,c) is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Note For $w = f(x,y,z)$, we also get

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

where the differentials dx, dy, dz, dw represent small changes or errors in x, y, z, w.

Ex Find an equation of the tangent plane to the surface $z = \frac{y-1}{x+1}$ at $(0, 0, -1)$

Sol The surface is the graph of $f(x,y) = \frac{y-1}{x+1}$.

The tangent plane at $(0,0,-1)$ is given by

$$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$= -1 + f_x(0,0)\,x + f_y(0,0)\,y.$$

$$f_x = \underbrace{\frac{\partial}{\partial x}}_{y \text{ constant}}\left(\frac{y-1}{x+1}\right) = (y-1)\frac{\partial}{\partial x}\left(\frac{1}{x+1}\right) = -\frac{y-1}{(x+1)^2}.$$

$$\Rightarrow f_x(0,0) = -\frac{0-1}{(0+1)^2} = 1.$$

$$f_y = \underbrace{\frac{\partial}{\partial y}}_{x \text{ constant}}\left(\frac{y-1}{x+1}\right) = \frac{1}{x+1}\frac{\partial}{\partial y}(y-1) = \frac{1}{x+1}.$$

$$\Rightarrow f_y(0,0) = \frac{1}{0+1} = 1.$$

The tangent plane at $(0,0,-1)$ is given by

$$\boxed{z = -1 + x + y}$$

Ex For $g(x,y) = x\sin(x+2y)$, estimate $g(2.1,-0.9)$ using the linear approximation near $(2,-1)$.

Sol $g_x = \frac{\partial}{\partial x}(x\sin(x+2y))$

$$= \frac{\partial}{\partial x}(x)\sin(x+2y) + x\frac{\partial}{\partial x}(\sin(x+2y))$$

product rule

$$= \sin(x+2y) + x\cos(x+2y)$$

$$\Rightarrow g_x(2,-1) = \sin(2-2\cdot1) + 2\cos(2-2\cdot1) = 2.$$

$$g_y = \frac{\partial}{\partial y}(x\sin(x+2y)) = x\frac{\partial}{\partial y}(\sin(x+2y))$$

$$= x\cos(x+2y)\frac{\partial}{\partial y}(x+2y) = 2x\cos(x+2y)$$

$$\Rightarrow g_y(2,-1) = 2\cdot2\cos(2-2\cdot1) = 4.$$

$$g(2,-1) = 2\sin(2-2\cdot1) = 0$$

The linear approximation near $(2,-1)$ is

$$g(x,y) \approx g(2,-1) + g_x(2,-1)(x-2) + g_y(2,-1)(y+1)$$

$$= 0 + 2(x-2) + 4(y+1)$$

$$\Rightarrow g(2.1,-0.9) \approx 0 + 2(2.1-2) + 4(-0.9+1) = \boxed{0.6}$$

Note The actual value is $g(2.1,-0.9) = 0.591\cdots$

Ex The body mass index of a person with height h (in meters) and weight m (in kilograms) is $B(m,h) = \frac{m}{h^2}$. The height is measured as 2m with a possible error of ± 0.02m, while the weight is measured as 100 kg with a possible error of ± 0.1 kg. Estimate the maximum error in the calculated body mass index.

Sol The errors are represented by differentials.

$$dB = \frac{\partial B}{\partial m} dm + \frac{\partial B}{\partial h} dh$$

$$\frac{\partial B}{\partial m} = \frac{\partial}{\partial m}\left(\frac{m}{h^2}\right) = \frac{1}{h^2}, \quad \frac{\partial B}{\partial h} = \frac{\partial}{\partial h}\left(\frac{m}{h^2}\right) = -\frac{2m}{h^3}.$$

$$m = 100,\ h = 2 \Rightarrow \frac{\partial B}{\partial m} = \frac{1}{4},\ \frac{\partial B}{\partial h} = -25.$$

$$\Rightarrow dB = \frac{1}{4} dm - 25 dh$$

The maximum is given by $dm = 0.1$, $dh = -0.02$.

$\Rightarrow$ The maximum error in the body max index is

$$\frac{1}{4} \cdot 0.1 - 25 \cdot (-0.02) = \boxed{0.525\ (kg/m^2)}$$